

# Non-Hermitian Tridiagonal Random Matrices and Returns to the Origin of a Random Walk

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We study a class of tridiagonal matrix models, the “ $q$ -roots of unity” models, which includes the sign ( $q=2$ ) and the clock ( $q=\infty$ ) models by Feinberg and Zee. We find that the eigenvalue densities are bounded by and have the symmetries of the regular polygon with  $2q$  sides, in the complex plane. Furthermore, the averaged traces of  $M^k$  are integers that count closed random walks on the line such that each site is visited a number of times multiple of  $q$ . We obtain an explicit evaluation for them.

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**KEY WORDS:** Band random matrices; non-hermetian random matrices; random walks.

## 1. INTRODUCTION

Random matrix ensembles are extensively studied since the early works of Wigner and Dyson, as effective models for the description of statistical properties of spectra of complex physical systems, which include resonances of heavy nuclei, quantum billiards, mesoscopic transport and quenched QCD.<sup>(1-4)</sup>

Tridiagonal matrices with random entries naturally occur in the simplest models of disordered one-dimensional crystals, beginning with the works by Dyson<sup>(5)</sup> and Schmidt.<sup>(6)</sup> If the spring constants are fixed while the masses are random, one is led to a tridiagonal Hermitian matrix with random entries on the main diagonal, often referred to as a random site problem.

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More recently, several authors studied the density of eigenvalues of non-Hermitian tridiagonal matrices still having the random entries in the main diagonal, after the model introduced by Hatano and Nelson<sup>(7)</sup> to describe the motion of vortices pinned by columnar defects in a superconductor. In this model, the spectral density was obtained only in the case of a Cauchy probability distribution for the independent random entries:<sup>(8, 10)</sup> in the large- $N$  limit, the eigenvalues lie on a “squeezed ellipse” in the complex plane with the addition of two “wings” on the real axis, which appear provided the strength of the random site entries exceeds a critical value. It was known for a long time<sup>(11, 12)</sup> that complex non-Hermitian random matrices may have eigenvalues filling a two-dimensional area in the complex plane. For a class of tridiagonal non-Hermitian models, with random hopping and random sites, Goldsheid and Khoruzhenko<sup>(9)</sup> found conditions for the probability distributions, such that in the large- $N$  limit the eigenvalues converge to a curve in the complex plane. Yet, in more general settings it seems still impossible to predict whether a non-Hermitian ensemble of random matrices has eigenvalues converging to a curve or filling a two-dimensional area or a fractal. These difficulties suggest the usefulness of investigating different types of tridiagonal non-Hermitian random matrices, possibly by a variety of techniques, even for models not directly related to a physics problem. Feinberg and Zee<sup>(13)</sup> considered a class of non-Hermitian random hopping models, with eigenvalue equation

$$t_{j-1}\psi_{j-1} + s_j^*\psi_{j+1} = E\psi_j \quad (1.1)$$

and given probability distributions for the hopping amplitudes  $t_j, s_j$ . In their first model, called the “clock model,” the hopping amplitudes are independent random phases. They found, numerically, that the eigenvalues are distributed in a disk in the complex plane, centered at the origin, with rotational invariance. In a second model, called the “sign model,” hopping amplitudes are independent and randomly equal to  $\pm 1$ . In the case of open chain, it is easy to see that the eigenvalues are functions of the  $N-1$  products  $t_j s_j^*$ , therefore, for the open chain, the distribution of eigenvalues of the “sign model” is also obtained by the study of eigenvalues of the matrix  $M(x, 1)$ , given below in Eq. (2.1), where the  $N-1$  independent variables  $x_i$  are independent and randomly equal to  $\pm 1$ .

In this paper we study a more general model, the “ $q$ -roots of unity” model, where each independent variable  $x_i$  is one of the  $q$  roots of unity, with uniform probability. Of course, for  $q=2$ , it is again the “sign model.”

For any  $q$ , the sum of the moduli of the two non-zero entries in any row of the matrix  $M(x, 1)$  equals two, therefore by Gershgorin circle theorem, all eigenvalues are in the disk  $|E| \leq 2$ . We find that for any  $N$  the

eigenvalues of the “ $q$ -roots of unity” model are inside the regular polygon with  $2q$  sides. The derivation of this very unusual boundary, together with the symmetry properties of the eigenvalue distribution are of intrinsic interest and are given in Section 2. In the limit  $q \rightarrow \infty$ , which corresponds to assuming the random variables  $x_i$  to be random phases, the support is the disk centered in the origin, with radius equal two and the density only depends on  $|E|$ . This case is the “clock model” of ref. 13. Our second result is the evaluation of the moments  $\langle \text{tr } M^n \rangle$  by mapping it into the problem of counting the returns to the origin of a restricted class of one-dimensional random walks, in Section 3.<sup>(14)</sup> The number of random walks in one dimension that originate at a given point, which we call the origin, and after  $2n$  random steps of unit length to the right or to the left, return to the origin (not necessarily for the first time) is  $(2n)!/(n!)^2$ . However, if we select among them the walks where each site different from the origin is visited an even number of times, the walks have to consist of a number of steps multiple of 4 and their number is smaller. Let us call such random walks **the even-visiting walks**. In the limit  $N \rightarrow \infty$ , the number of even-visiting walks provides the moments  $\langle \text{tr } M^n \rangle$  of the “sign model.” In the same way, the number of returns to the origin of one dimensional walks where each site of the walk is visited a number of times multiple of  $q$  provides the moments  $\langle \text{tr } M^n \rangle$  of the “ $q$ -roots of unity” model.

As we mentioned, we do not know physics models related to the “sign model.” It may then be proper to spend a few words to justify the present investigation. The evaluation of moments  $\langle \text{tr } M^n \rangle$ , where the tridiagonal matrix  $M(x, 1)$  is given in Eq. (2.1), is conveniently mapped into a random walk problem in one dimension, independently of the probability densities of the random variables  $x_i$ . Furthermore, provided the  $x_i$  are real independent variables and the probability densities are even functions, only the even-visiting walks are relevant. However, only for the “sign model” the evaluation of the moments  $\langle \text{tr } M^n \rangle$  corresponds to counting the even-visiting walks, whereas if a different even probability density is considered, to each even-visiting walk one has to assign a weight.

Our main result, for the “sign model” is

$$\lim_{N \rightarrow \infty} \frac{1}{N} \langle \text{tr } M^{4k} \rangle = \sum_{t=1}^k \sum_{\{n_i\}} S_{[n_1, n_2, \dots, n_t]} \tag{1.2}$$

where the sum  $\sum_{\{n_i\}}$  is over the  $t$  positive integers  $n_i$  with the restriction  $n_1 + n_2 + \dots + n_t = k$  and  $S_{[n_1, n_2, \dots, n_t]}$  is the number of the relevant walks with “width”  $w = t$ :

$$S_{[n_1, n_2, \dots, n_t]} = \frac{2k}{n_1} \prod_{i=1}^{t-1} \binom{2n_{i+1} + 2n_i - 1}{2n_{i+1}} \tag{1.6}$$

A very similar result, for arbitrary value of  $q$ , for the “ $q$ -roots of unity” model, is obtained in Section 3.

As this work was completed, a new work appeared<sup>(15)</sup> where the spectral density of the “clock model” was obtained. Since our work does not provide quantitative information on spectral densities, it has little overlap with it. However several very interesting assertions, like the analysis of the two non-linear maps  $T_+$ ,  $T_-$ , and the existence of bound states, discussed for the “clock model,” are equally valid for the “sign model.”

## 2. THE $q$ -ROOTS OF UNITY MODEL

We consider the ensemble  $\mathcal{E}_N(q, 1)$  of tridiagonal random matrices of size  $N \times N$ ,

$$M(x, 1) = \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & x_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & x_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & x_4 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & x_{N-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad (2.1)$$

where the random entries  $x_i$ ,  $i = 1, 2, \dots, N-1$ , are  $q$ -roots of unity:  $x_i^q = 1$ . More precisely, they are independent and identically distributed random variables, with uniform probability distribution over the set of the  $q$ -roots of unity:

$$P(x) = \frac{1}{q} \sum_{j=0}^{q-1} \delta(x - w_q^j), \quad w_q \equiv e^{i(2\pi/q)} \quad (2.2)$$

The ensemble consists of  $q^{N-1}$  different matrices.

For later convenience, we introduce the notation  $M(x, y)$  to denote a general tridiagonal matrix  $M$  of order  $N$  with the upper and lower diagonals specified by vectors  $x = \{x_1, \dots, x_{N-1}\}$  and  $y = \{y_1, \dots, y_{N-1}\}$ .

In this section we are interested in the investigation of the symmetries and boundaries of the eigenvalue distribution of the ensemble, in the large  $N$  limit. We consider, for finite  $N$ , the set  $\sigma(q, N)$  in the complex plane of all the eigenvalues of matrices belonging to the ensemble and show that it has nice symmetry properties.

**Proposition 1.** The set  $\sigma(q, N)$  is invariant under the transformations

$$(i) \quad E \rightarrow -E, \quad (ii) \quad E \rightarrow E^*, \quad (iii) \quad E \rightarrow E^{i(\pi/q)}$$

*Proof.* The characteristic polynomial of a generic tridiagonal matrix  $M(x, y)$  only depends on the products  $x_i y_i, i = 1, 2, \dots, N - 1$ . Therefore, for a given matrix  $M(x, 1)$  in the ensemble we introduce the two matrices  $M(z, z)$  and  $M(-z, -z)$  where  $z_i^2 = x_i$ . Clearly they have opposite eigenvalues, meaning that both  $E$  and  $-E$  are eigenvalues of  $M(x, 1)$ .

The ensemble  $\mathcal{E}_N(q, 1)$  is closed under complex conjugation, and if  $\{E_i\}$  are the eigenvalues of  $M(x, 1)$ ,  $\{E_i^*\}$  are the eigenvalues of  $M(x^*, 1)$ .

The set of matrices  $M(z, z)$ , where each  $z_i$  is randomly chosen in the set of the  $2q$ -roots of unity, is invariant under multiplication of a matrix by the scalar  $e^{i\pi/q}$ . Therefore, if  $\{E_i\}$  are the eigenvalues of  $M(x, 1)$ ,  $\{E_i e^{i\pi/q}\}$  are the eigenvalues of  $M(xw_q, 1)$ .

We conclude that the set  $\sigma(q, N)$  has the above stated symmetries (note that invariance i follows from iii). ■

**Proposition 2.** The set  $\sigma(q, N)$  is contained in the regular  $2q$ -polygon in the complex  $E$ -plane, with corners  $E_k = 2e^{ik\pi/q}, k = 0, 1, \dots, 2q - 1$ .

*Proof.* Let  $E$  be an eigenvalue of a given matrix  $M(x, 1)$  in the ensemble; because of the symmetry, we assume that  $0 \leq \arg E \leq \pi/q$ . The eigenvalue equation for  $M(x, 1)$  is  $u_{i-1} + x_i u_{i+1} = E u_i$ , with  $u_0 = u_{N+1} = 0$ . Let  $I_k$  be the set of integers  $j = 1, \dots, N - 1$  such that  $x_j = w_q^k$ . After multiplication by  $u_i^*$  of the eigenvalue equation and summation over  $i$ :

$$\begin{aligned} E &= \sum_{i=1}^N u_i^* u_{i-1} + \sum_{i=1}^N x_i u_i^* u_{i+1} \\ &= \sum_{i=1}^N u_{i+1}^* u_i + \sum_{k=1}^q w_q^k \sum_{i \in I_k} u_i^* u_{i+1} \\ &= \sum_{k=1}^q \sum_{i \in I_k} u_{i+1}^* u_i + w_q^k u_i^* u_{i+1} \\ &= \sum_{k=1}^q e^{i(k\pi/q)} \sum_{i \in I_k} (u_i^* u_{i-1} e^{-ik(\pi/q)} + c.c.) \end{aligned}$$

We finally take the real and imaginary parts of  $E$  and build the majorization:

$$\begin{aligned}
 & \operatorname{Re} E \cos\left(\frac{\pi}{2q}\right) + \operatorname{Im} E \sin\left(\frac{\pi}{2q}\right) \\
 &= \sum_{k=1}^q \cos\left(k\frac{\pi}{q} - \frac{\pi}{2q}\right) \sum_{i \in I_k} (u_i^* u_{i-1} e^{-ik\pi/q} + c.c.) \\
 &\leq 2 \sum_{k=1}^q \left| \cos\left(k\frac{\pi}{q} - \frac{\pi}{2q}\right) \right| \sum_{i \in I_k} |u_i^* u_{i-1}| \\
 &\leq 2 \cos\left(\frac{\pi}{2q}\right) \sum_{k=1}^q \sum_{i \in I_k} |u_i^* u_{i-1}| \\
 &\leq 2 \cos\left(\frac{\pi}{2q}\right)
 \end{aligned}$$

Therefore:  $\operatorname{Im} E \leq \cotg(\pi/2q)(2 - \operatorname{Re} E)$ , which ends the proof.  $\blacksquare$

From the discussion of Proposition 1, we conclude that the distribution of the eigenvalues  $\rho(E, E^*)$  for the ensemble  $\mathcal{E}_N(q, 1)$  has the listed symmetries, and therefore it is a symmetric function of the variables  $E^{2q}$  and  $(E^*)^{2q}$ . This puts constraints on the moments.

**Proposition 3.**  $\langle E^m E^{*n} \rangle \neq 0$  only if  $m+n$  is even and  $|m-n| = 2rq$ .

*Proof.* Since the expectation value is symmetric in  $m$  and  $n$ , we put  $m \geq n$ . Due to the symmetry  $E \rightarrow -E$  of the density, non-vanishing of the expectation value requires  $m+n=2s$ , so that  $m-n=2\ell$ . By the  $2q$ -fold symmetry of the density and the support  $\sigma$ , we may evaluate the expectation by integrating over  $\omega$ , the angular sector  $0 \leq \arg E \leq \pi/q$  of  $\sigma$ :

$$\langle |E|^{2s} E^\ell (E^*)^{-\ell} \rangle = \int_{\omega} d^2E \rho(E, E^*) |E|^{2s} \sum_{k=0}^{2q-1} (E e^{ik\pi/q})^\ell (E^* e^{-ik\pi/q})^{-\ell}$$

We get a non-zero result for the sum only if  $\ell = rq$ .  $\blacksquare$

In the limit  $q \rightarrow \infty$ , where each number  $x_i$  is an arbitrary number on the unit circle, the set  $\sigma(\infty, N)$  is invariant under complex rotations, and the eigenvalue density is a function of  $|E|$ .

### 3. THE MOMENTS AND COUNTING THE RETURNS OF THE RELEVANT WALKS

In this section we study a set of moments of the probability density  $\rho(E, E^*)$  of the “q-roots of unity” model, which have the interesting interpretation as counting numbers of the random walks on the line that return to the origin and visit each intermediate site a number of times which is multiple of  $q$ . These moments have generating function

$$G(z) = \frac{1}{N} \left\langle \text{tr} \frac{1}{z - M} \right\rangle \tag{3.1}$$

where the expectation value is evaluated on the independent identically distributed random variables  $x_i$ :

$$\langle f(M) \rangle = \int \prod_{i=1}^N P(x_i) dx_i f(M(x, 1)) \tag{3.2}$$

The formal perturbative expansion of  $G(z)$  is

$$G(z) = \frac{1}{N} \sum_{k=0}^{\infty} \frac{\langle \text{tr} M^k \rangle}{z^{k+1}} \tag{3.3}$$

Since we are interested in the limit  $N \rightarrow \infty$ , any term on the diagonal  $\langle [M^k]_{rr} \rangle$  has the same value, and the trace merely cancels the  $1/N$  factor.

It is useful to consider the one to one map between the non vanishing terms contributing to  $[M^k]_{ab}$  and the random walks in one dimension which originate at site  $a$  to arrive at site  $b$  after  $k$  steps.

Let us consider the term  $k = 8$

$$\sum_{a, b, c, d, e, f, g} \langle M_{ra} M_{ab} M_{bc} M_{cd} M_{de} M_{ef} M_{fg} M_{gr} \rangle \tag{3.4}$$

By recalling that the non zero matrix elements are  $M_{ij} = 1$  if  $j = i - 1$ , and  $M_{ij} = x_i$  if  $j = i + 1$ , each term in the sum (3.4) corresponds to a walk of 8 steps, originating and ending at site  $r$ , with 4 steps up and 4 steps down. For instance, the sequence  $M_{r, r+1} M_{r+1, r+2} M_{r+2, r+1} M_{r+1, r+2} M_{r+2, r+1} M_{r+1, r} M_{r, r+1} M_{r+1, r} = x_r \cdot x_{r+1} \cdot 1 \cdot x_{r+1} \cdot 1 \cdot 1 \cdot x_r \cdot 1 = 1$  is shown in Fig. 1, while the sequence  $x_r \cdot x_{r+1} \cdot 1 \cdot 1 \cdot x_r \cdot 1 \cdot x_r \cdot 1 = x_r \cdot x_{r+1}$  is shown in Fig. 2. Each walk corresponding to a product of random variables  $\prod_j (x_j)^{n_j}$  where all the powers  $n_j$  are multiple of  $q$  yields a contribution  $+1$ , while the walks where at least one power  $n_j$  is not a multiple of  $q$  are averaged to zero, therefore being a class of irrelevant walks. Then  $(1/N) \langle \text{tr} M^k \rangle$  is equal to

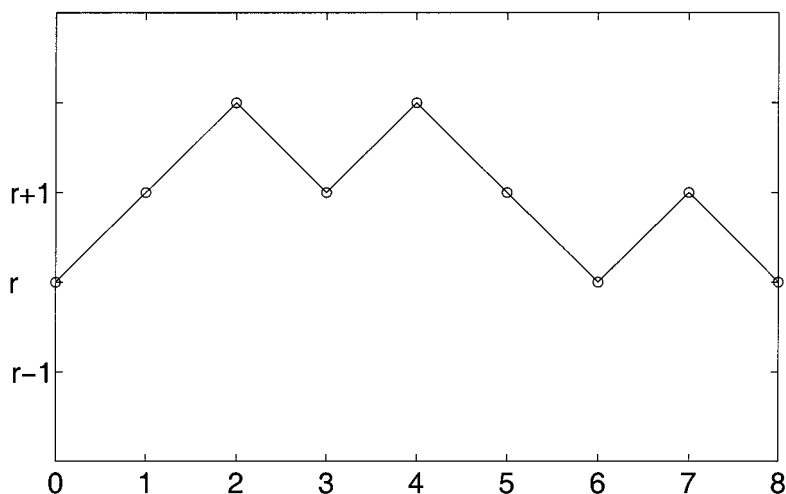


Fig. 1. One of the even-visiting random walks, returning to site  $r$  after 8 steps, with width  $w = 2$ .

the number of relevant walks of  $k$  steps, that is the walks from a fixed site  $r$  to the same site  $r$ , such that each intermediate site is visited a multiple of  $q$  times (see Fig. 3).

Because the number of steps up (each associated to a random variable  $x_i$ ) is equal to number of steps down (each associated to a factor one), the

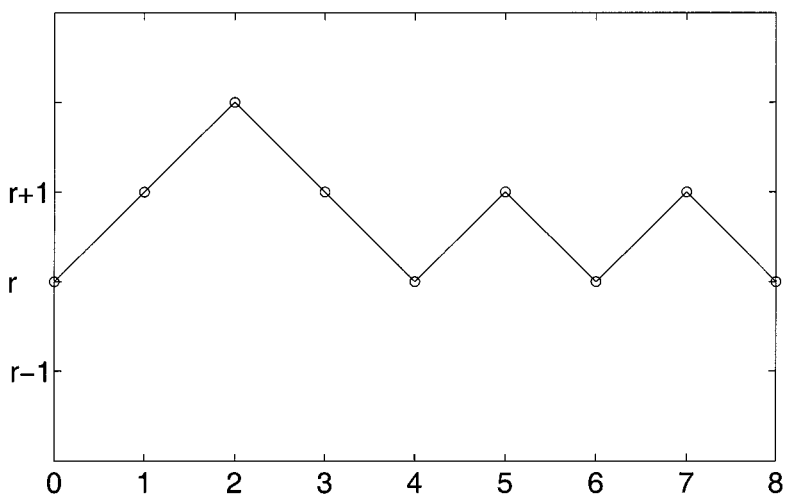


Fig. 2. One of the random walks not belonging to the class of even-visiting random walks.



total number of steps  $k$  of the relevant walks is a multiple of  $2q$  and we rewrite Eq. (3.3) as

$$G(z) = \sum_{k=0}^{\infty} \frac{c_k}{z^{2qk+1}}; \quad c_k = \lim_{N \rightarrow \infty} \frac{1}{N} \langle \text{tr } M^{2qk} \rangle \quad (3.5)$$

This power series is absolutely convergent for  $|z| > 2$ . In the case of  $q = \infty$  we obtain

$$G(z) = \frac{1}{z}, \quad \text{for } |z| > 2 \quad (3.6)$$

Next we present the combinatorial evaluation of the coefficients  $c_k$  for the case  $q=2$ , that is the number of returns to the origin for even visiting walks. The case of general integer values of  $q$  requires only trivial generalization, presented in the following paragraph.

*Counting the Returns to the Origin of Even Visiting Walks.* We shall indicate with  $N(2n_{-s}, 2n_{-s+1}, \dots, 2n_{-1}; 2n_0, 2n_1, \dots, 2n_j)$  the number

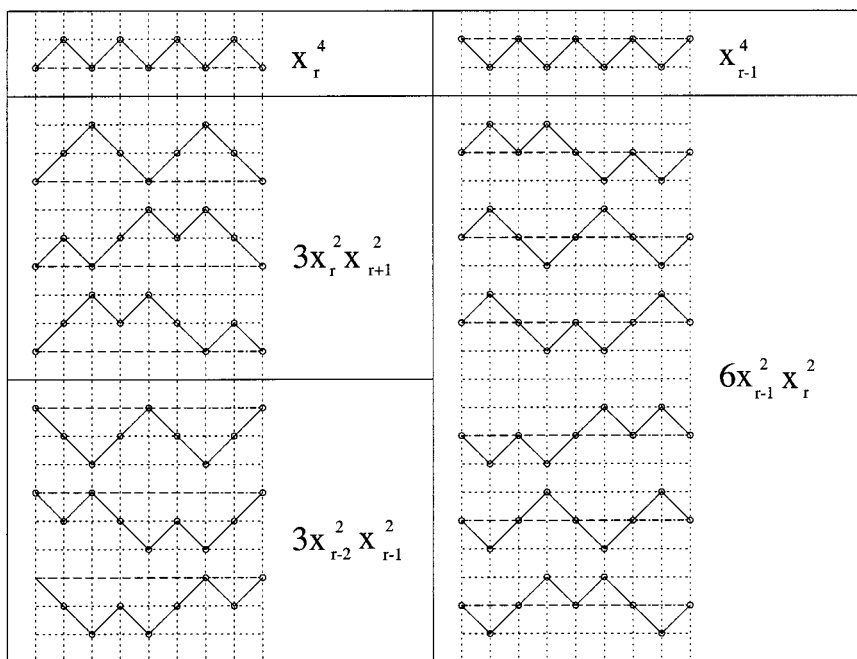


Fig. 3. Here are shown the 14 paths of 8 steps, relevant for the “sign model.”

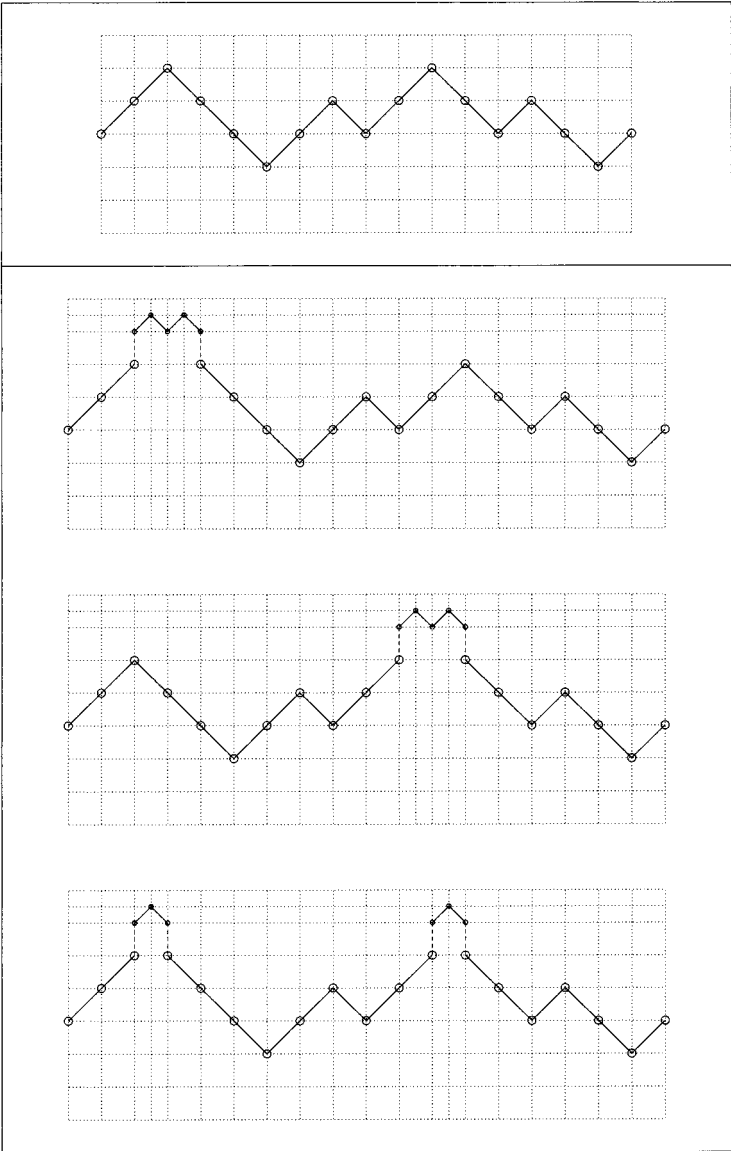


Fig. 4. The figure illustrates the insertion of a 4-steps part over anyone of the tops of a previous relevant walk of width  $w$ , thus obtaining new relevant walks, of width  $w+1$ , according to Eq. (3.7).

of even visiting walks from site  $r$  to site  $r$  corresponding to the product  $(x_{r-s})^{2n-s} \dots (x_{r-1})^{2n-1} (x_r)^{2n_0} (x_{r+1})^{2n_1} \dots (x_{r+j})^{2n_j}$ . All the integers  $n_{-s}, \dots, n_j$  are positive; the semicolon separates the exponents  $n_i$  associated to sites  $i < r$  from those associated to sites  $i \geq r$ . We omit strings of zeros external to the string of positive integers, just keeping one zero for walks visiting only sites  $s \geq r$  or only sites  $s \leq r$ . In the first case we write the multiplicity as  $N(0; 2n_0, \dots, 2n_j)$ , in the latter case we write  $N(2n_{-s}, \dots, 2n_{-1}; 0)$ . It is easiest to begin with the evaluation of  $N(0; 2n_0, \dots, 2n_j)$ . The length of the walk is  $4(n_0 + n_1 + \dots + n_j)$ . The ‘‘maximum site’’ visited is the site  $r + j + 1$ , visited  $2n_j$  times; the ‘‘minimum site’’ visited is the site  $r$ , visited  $2n_0 + 1$  times.

The number  $N(0; 2n_0, 2n_1, \dots, 2n_j, 2n_{j+1})$  is related to  $N(0; 2n_0, 2n_1, \dots, 2n_j)$  in the following way: new walks of length two corresponding to  $(x_{r+j+1} \cdot 1)$  may be inserted in each of the maxima of the previous walk (see Fig. 4). Since  $2n_{j+1}$  identical objects are placed in  $2n_j$  places in  $\binom{2n_{j+1} + 2n_j - 1}{2n_{j+1}}$  ways, we obtain

$$\begin{aligned}
 N(0; 2n_0, 2n_1, \dots, 2n_j, 2n_{j+1}) \\
 = \binom{2n_{j+1} + 2n_j - 1}{2n_{j+1}} N(0; 2n_0, 2n_1, \dots, 2n_j)
 \end{aligned}
 \tag{3.7}$$

By iterating Eq. (3.7) with the initial condition  $N(0; 2n_0) = 1$  one obtains

$$N(0; 2n_0, 2n_1, \dots, 2n_j) = \prod_{i=0}^{j-1} \binom{2n_{i+1} + 2n_i - 1}{2n_{i+1}}
 \tag{3.8}$$

We proceed to evaluate the multiplicity of the relevant random walks that visit sites  $r, r + 1, \dots, r + j$  as often as before, and in addition visit  $2n_{-1}$  times the site  $r - 1$ . Each walk of this class may be obtained by inserting  $2n_{-1}$  walks of length two  $(1 \cdot x_{r-1})$  in each of the  $2n_0 + 1$  minima of the walk of the previous class. Therefore

$$\begin{aligned}
 N(2n_{-1}; 2n_0, 2n_1, \dots, 2n_j) \\
 = \binom{2n_{-1} + 2n_0}{2n_{-1}} N(0; 2n_0, 2n_1, \dots, 2n_j)
 \end{aligned}
 \tag{3.9}$$

The procedure may be repeated to include the relevant walks which visit sites  $r - 2, r - 3, \dots, r - s$ . We obtain

$$\begin{aligned}
 & N(2n_{-s}, 2n_{-s+1}, \dots, 2n_{-1}; 2n_0, 2n_1, \dots, 2n_j) \\
 &= \left[ \prod_{p=0}^{s-2} \binom{2n_{-s+p} + 2n_{-s+p+1} - 1}{2n_{-s+p}} \right] \binom{2n_{-1} + 2n_0}{2n_{-1}} \left[ \prod_{i=0}^{j-1} \binom{2n_{i+1} + 2n_i - 1}{2n_{i+1}} \right] \tag{3.10}
 \end{aligned}$$

The coefficient  $c_k$  is the number of the even visiting walks of  $4k$  steps and it is the sum of the multiplicities  $N(2n_{-s}, 2n_{-s+1}, \dots, 2n_{-1}; 2n_0, 2n_1, \dots, 2n_j)$  given above, where  $k = n_{-s} + n_{-s+1} + \dots + n_j$ .

The evaluation may be somewhat simplified, by considering walks of fixed width, that is the difference between the ‘‘maximum site’’ visited and the ‘‘minimum site’’ visited. We consider the set of ordered partitions of  $k$  into positive integers  $[n_1, n_2, \dots, n_t]$  where  $k = \sum n_p$ . Each ordered sequence,  $[n_1, n_2, \dots, n_t]$ , corresponds to  $t + 1$  classes of even visiting walks, which are associated to the products

$$\begin{aligned}
 & (x_{r-t})^{2n_1} (x_{r-t+1})^{2n_2} \dots (x_{r-1})^{2n_t}; \\
 & (x_{r-t+1})^{2n_1} (x_{r-t+2})^{2n_2} \dots (x_r)^{2n_t}; \\
 & \dots \quad \dots \quad \dots \quad \dots; \\
 & (x_r)^{2n_1} (x_{r+1})^{2n_2} \dots (x_{r+t-1})^{2n_t}
 \end{aligned} \tag{3.11}$$

All walks in Eq. (3.11) have the same width  $w = t$ . Their multiplicities, given in Eq. (3.10) are simply related and their sum is

$$S_{[n_1, n_2, \dots, n_t]} = \frac{2k}{n_1} \prod_{i=1}^{t-1} \binom{2n_{i+1} + 2n_i - 1}{2n_{i+1}}; \quad S_{[n_1]} = 2 \tag{3.12}$$

Next we sum over the ordered partitions  $[n_1, n_2, \dots, n_t]$  of  $k$  into  $t$  parts and finally over the different widths, from 1 to  $k$

$$c_k = \sum_{t=1}^k \sum_{\{n_i\}} S_{[n_1, n_2, \dots, n_t]} \tag{3.13}$$

where the sum  $\sum_{\{n_i\}}$  is over the  $t$  positive integers  $n_i$  with the restriction  $n_1 + n_2 + \dots + n_t = k$ . The evaluation of Eq. (3.13) may be automated and we find the first coefficients  $c_k$ :

|          |     |     |     |     |     |     |
|----------|-----|-----|-----|-----|-----|-----|
| $c_0$    |     |     |     |     |     | 1   |
| $c_1$    |     |     |     |     |     | 2   |
| $c_2$    |     |     |     |     |     | 14  |
| $c_3$    |     |     |     |     |     | 116 |
| $c_4$    |     |     |     | 1   |     | 110 |
| $c_5$    |     |     |     | 11  |     | 372 |
| $c_6$    |     |     |     | 123 |     | 020 |
| $c_7$    |     |     | 1   | 384 |     | 168 |
| $c_8$    |     |     | 16  | 058 |     | 982 |
| $c_9$    |     |     | 190 | 948 |     | 796 |
| $c_{10}$ |     | 2   | 317 | 085 |     | 924 |
| $c_{11}$ |     | 28  | 602 | 719 |     | 576 |
| $c_{12}$ |     | 358 | 298 | 116 |     | 092 |
| $c_{13}$ | 4   | 545 | 807 | 497 |     | 272 |
| $c_{14}$ | 58  | 321 | 701 | 832 |     | 408 |
| $c_{15}$ | 755 | 700 | 271 | 652 |     | 816 |
| $c_{16}$ | 9   | 878 | 971 | 460 | 641 | 414 |

(3.14)

The ratios  $c_n/c_{n-1}$  rise monotonically with a rate slower at higher values of  $n$ . We know from the previous section that the eigenvalues of the random matrix  $M$  are inside the square with vertices in the four points  $\pm 2, \pm 2i$ , it then follows that  $c_n \sim a^n$ , with  $a \leq 16$ . The plot of the coefficients  $c_n$  for  $n=4$  to 19 in Fig. 5 is consistent with the expected asymptotics. ■

*Counting the Returns to the Origin of Walks for Generic  $q$ .* The combinatorial evaluation given above holds with trivial modifications for the “q-roots of unity” model, described in Eqs. (2.1), (2.2). Again  $\text{tr } M^k$  corresponds to one-dimensional walks returning to the original site after  $k$  steps, each walk being the product of  $k/2$  random variables  $x_i$ . The average of each product, with the factorized probability distribution (2.2) vanishes unless each random variable occurs with a power multiple of  $q$ , in this case the product has average one. Therefore  $k/2$  is a multiple of  $q$  and the class of relevant walks is such that each site (apart the origin of the walk) is visited a number of times multiple of  $q$ . Let  $N(0; qn_0, qn_1, \dots, qn_j)$  be the number of the relevant walks corresponding to the multiplicity of the product  $(x_r)^{qn_0} (x_{r+1})^{qn_1} \dots (x_{r+j})^{qn_j}$ . The insertion above the top sites or below the bottom ones proceeds as before leading to a trivial generalization for the number of relevant walks  $S_{[n_1, n_2, \dots, n_t]}$  having fixed width  $t$

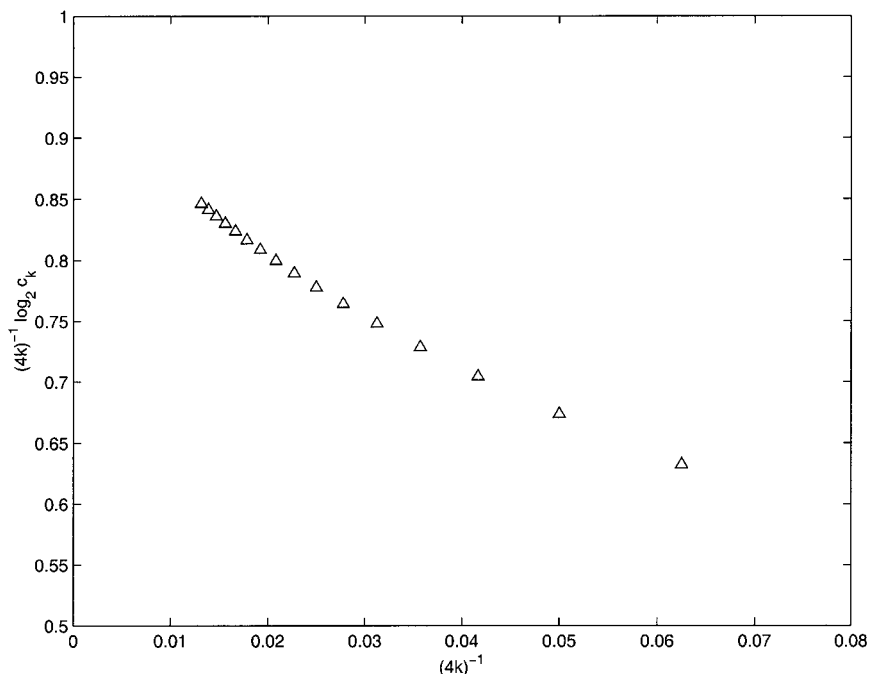


Fig. 5. The triangles are the values of  $(1/4k) \log_2 c_k$  versus  $(1/4k)$ , for  $k=4$  to 19. The convergence for large values of  $k$  is consistent with  $c_k \sim a^k$ , with  $a \leq 16$ .

$$S_{[n_1, n_2, \dots, n_t]} = \frac{2k}{n_1} \prod_{i=1}^{t-1} \binom{qn_{i+1} + qn_i - 1}{qn_{i+1}} \quad (3.15)$$

$$c_k = \sum_{t=1}^k \sum_{\{n_i\}} S_{[n_1, n_2, \dots, n_t]} \quad (3.16)$$

where the sum  $\sum_{\{n_i\}}$  is over the  $t$  positive integers  $n_i$  with the restriction  $n_1 + n_2 + \dots + n_t = k$ . ■

Let us close with few words for the special value  $q=1$ . In this case the matrix  $M$  in Eq. (2.1) is no longer random, it is the real symmetric matrix with all  $x_i=1$ . Its spectral distribution is well known and the analysis in random walks is not necessary. Still one may find that  $c_k = \text{tr } M^k(1, 1) = \binom{2k}{k}$  is the number of one-dimensional random walks returning to the origin after  $2k$  steps. We checked, for several values of  $k$ , that Eqs. (3.15), (3.16) reproduce the correct value for  $q=1$ .

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